Chapter 3

Materials

We are now going to look at how fields interact with materials. Once we understand this, we can do a variety of useful things. For example, this interaction is how your microwave over heats food. It's also how lenses, polarizers, and wave rotators work in an optical system. We will find some very interesting behaviors of the waves in certain materials.

3.1 Classical Model of Fields in Materials

We begin by assuming that the nucleus of an atom is very massive. Therefore, if an electric field is applied to the atom, the nucleus will not be accelerated appreciably. The electrons, however, will move in response to this field. We can model this as a spring-mass system. The spring has a spring constant K, and there is a damping coefficient γ. The electron has a mass m and a charge q. For an applied field of $\overline{E} = \hat{x} E_o e^{j\omega t}$, the net force on the electron can be written as

$$F = -\gamma \dot{x} - Kx + E_o q e^{j\omega t} \tag{3.1}$$

where $x(t)$ is the position of the electron and the dot denotes time differentiation. Using Newton's second law of motion,

$$F = ma = m\ddot{x} \tag{3.2}$$

we obtain the differential equation

$$\ddot{x} + \frac{\gamma}{m}\dot{x} + \frac{K}{m}x = \frac{q}{m}E_o e^{j\omega t} \tag{3.3}$$

$$\ddot{x} + 2\alpha\dot{x} + \omega_o^2 x = \frac{q}{m}E_o e^{j\omega t} \tag{3.4}$$

where $\alpha = \gamma/2m$, $\omega_o^2 = K/m$.

3.1.1 Complementary Solution

The complementary solution to (3.4) is obtained by setting the right hand side of the equation to zero.

$$\ddot{x}_c + 2\alpha\dot{x}_c + \omega_o^2 x_c = 0 \tag{3.5}$$

$$x_c(t) = Ae^{\beta_1 t} + Be^{\beta_2 t} \tag{3.6}$$

$$\beta_{1,2} = \frac{-2\alpha \pm \sqrt{4\alpha^2 - 4\omega_o^2}}{2} = -\alpha \pm \sqrt{\alpha^2 - \omega_o^2} \tag{3.7}$$

This represents the transient solution. It may oscillate if $\omega_o^2 > \alpha^2$, but it will decay as it oscillates.

3.1.2 Particular Solution

The particular solution to (3.4) represents the steady state solution. Let

$$x_p(t) = x_o e^{j\omega t} \tag{3.8}$$

We need to solve for x_o in terms of the parameters in the differential equation. Plugging this into the differential equation leads to

$$(-\omega^2 + j2\alpha\omega + \omega_o^2)x_o e^{j\omega t} = \frac{q}{m}E_o e^{j\omega t} \tag{3.9}$$

which can be solved to obtain

$$x_o = \frac{qE_o}{m}\frac{1}{(\omega_o^2 - \omega^2) + j2\alpha\omega} \tag{3.10}$$

3.1.3 Dielectric Model

We will now use this solution for the mass-spring model to derive the permittivity of a bulk material made up of many electrons with the same dynamical model. We define the Vector Dipole Moment for each electron relative to its rest position ($x_p = 0$) as

$$\overline{p}(t) = \hat{x}qx_p(t) \tag{3.11}$$

and the Polarization Vector or bulk dipole moment for many electrons undergoing similar motion as

$$\overline{P}(t) = N\overline{p}(t) = \hat{x}Nqx_p(t) \tag{3.12}$$

where N is the number of dipoles per unit volume. Also, from electromagnetic theory,

$$\overline{P}(t) = \epsilon_0\chi_e\overline{E}(t) \tag{3.13}$$

where χ_e is the electric susceptibility of the material ($\epsilon_r = 1 + \chi_e$). If we equate these two definitions for $\overline{P}(t)$ we obtain

$$\hat{x}Nqx_p(t) = \epsilon_0\chi_e\overline{E}(t) \tag{3.14}$$

$$\hat{x}Nq\frac{qE_o}{m}\frac{1}{(\omega_o^2 - \omega^2) + j2\alpha\omega}e^{j\omega t} = \hat{x}\epsilon_0\chi_e E_o e^{j\omega t} \tag{3.15}$$

$$\chi_e = \frac{Nq^2}{m\epsilon_0}\frac{1}{(\omega_o^2 - \omega^2) + j2\alpha\omega} \tag{3.16}$$

$$\epsilon_r = 1 + \frac{Nq^2}{m\epsilon_0}\frac{1}{(\omega_o^2 - \omega^2) + j2\alpha\omega} \tag{3.17}$$

This simple model for the permittivity of a dielectric material predicts behaviors like:

1. Loss: due to imaginary part of ϵ_r
2. Dispersion: dependence of ϵ_r on frequency
3. Anisotropy: ϵ_r can be different for different polarizations if the spring constant and damping coefficient are different for these different polarizations.
4. Nonlinearity: ϵ_r depends on the magnitude of $\overline{E}$

Our main focus in this class is Item #3.

3.2 Anisotropic Media

In general, anisotropic media are characterized by a permittivity *tensor* given as

$$\overline{\overline{\epsilon}} = \begin{bmatrix} \epsilon_{xx} & \epsilon_{xy} & \epsilon_{xz} \\ \epsilon_{yx} & \epsilon_{yy} & \epsilon_{yz} \\ \epsilon_{zx} & \epsilon_{zy} & \epsilon_{zz} \end{bmatrix} \tag{3.18}$$

The components of the tensor $\overline{\overline{\epsilon}}$ have a simple physical interpretation: if an electric field in a given direction (say z) is applied, then $\chi_{e,xz} = \epsilon_{xz} - 1$ gives the component of the additional polarization vector $\overline{P}$ in the x direction according to Eq. (3.13), and so forth for the other components.

This tensor multiplication works like a matrix/vector multiplication ($\overline{D} = \overline{\overline{\epsilon}} \cdot \overline{E}$). In matrix form this would look like

$$\begin{bmatrix} D_x \\ D_y \\ D_z \end{bmatrix} = \begin{bmatrix} \epsilon_{xx} & \epsilon_{xy} & \epsilon_{xz} \\ \epsilon_{yx} & \epsilon_{yy} & \epsilon_{yz} \\ \epsilon_{zx} & \epsilon_{zy} & \epsilon_{zz} \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix} \tag{3.19}$$

For example, we can look at just the x-component as given by

$$D_x = \epsilon_{xx} E_x + \epsilon_{xy} E_y + \epsilon_{xz} E_z \tag{3.20}$$

For all lossless ($\sigma = 0$) media it is always possible to express the permittivity matrix as

$$[\epsilon] = \begin{bmatrix} \epsilon_{11} & j\epsilon_{12} & 0 \\ -j\epsilon_{12} & \epsilon_{22} & 0 \\ 0 & 0 & \epsilon_{33} \end{bmatrix}. \tag{3.21}$$

This transformation is accomplished by choosing the proper orientation of the axes (x, y, z). For us, this means that properly defining our coordinate frame can result in a diagonal permittivity tensor. This coordinate frame is called the *Principal System* with the coordinate axes called the *Principal Axes*.

The off-diagonal elements exist in gyrotropic materials. We will be discussing gyrotropic materials later. For now we are going to be looking at non-gyrotropic materials with zero off-diagonal elements.

In the principal system, the permittivity matrix is

$$\bar{\bar{\epsilon}} = \begin{bmatrix} \epsilon_x & 0 & 0 \\ 0 & \epsilon_y & 0 \\ 0 & 0 & \epsilon_z \end{bmatrix} \tag{3.22}$$

If $\epsilon_x \neq \epsilon_y \neq \epsilon_z$, the medium is biaxial. If any two are equal (for example $\epsilon_x = \epsilon_y \neq \epsilon_z$), the medium is uniaxial.

These principal dielectric constants can also be denoted using index of refraction as given by

$$n_x^2 = \frac{\epsilon_x}{\epsilon_o} \tag{3.23}$$

$$n_y^2 = \frac{\epsilon_y}{\epsilon_o} \tag{3.24}$$

$$n_z^2 = \frac{\epsilon_z}{\epsilon_o} \tag{3.25}$$

Recall that the relative permittivity of a material is related to how an externally applied electric field interacts with the electrons of atoms that compose the material. The anisotropic nature of a meterial is caused by symmetry of the atoms that compose the material. If there is no symmetry among the atoms then the material is isotropic. Thus, only crystals (or cystal like materials) are anisotropic. The crystal symmetry determines whether a material is uniaxial or biaxial. Likewise, the symmetry directions determine the principal axes of the material.

It is interesting to note that in general, $\overline{D}$ and $\overline{E}$ are no longer parallel. For example see Fig. 3.1.

3.2.1 General Characteristics of Plane Waves in Anisotropic Media

In Chapter 1, we demonstrated that for plane waves, we can replace the ∇ operator by $-j\overline{k}$. Therefore, we have

$$\nabla \cdot \overline{D} = -j\overline{k} \cdot \overline{D} = 0 \tag{3.26}$$

which means that $\overline{k} \perp \overline{D}$. Since $\overline{D}$ and $\overline{E}$ are no longer parallel in general, this means that the phase progression is perpendicular to $\overline{D}$ but not necessarily to $\overline{E}$. Furthermore, $\nabla \cdot \overline{B} = \mu \nabla \cdot \overline{H} = -j\overline{k} \cdot \overline{H} = 0$ so we know that $\overline{k} \perp \overline{H}$

Furthermore, since $\overline{D}$ is not necessarily parallel to $\overline{E}$, the Poynting vector $\overline{S}$ is not necessarily parallel to $\overline{k}$. So, power flow and phase progression occur in different directions.

We will restrict ourselves to propagation along one of the principal axes. This will significantly simplify our analysis. Under these conditions, we can define steps for analyzing the behavior of plane waves in anisotropic media:

1. Write Maxwell's equations in vector component form
2. Eliminate the magnetic field to obtain a relationship in terms of $\overline{E}$ only
3. Find the possible phase velocities
4. Find the polarization for each phase velocity (characteristic waves)

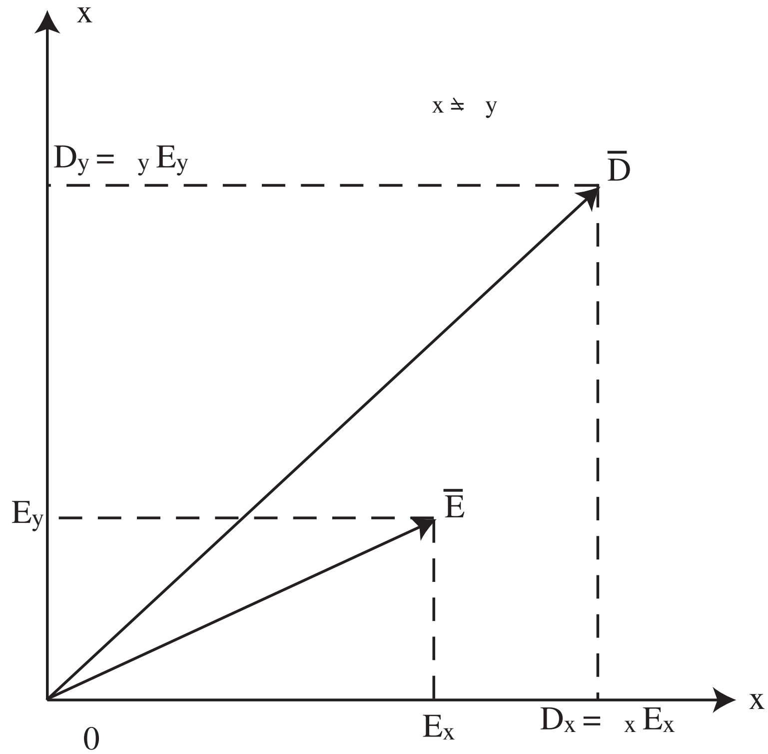

Figure 3.1: Representative $\overline{E}$ and $\overline{D}$ for an anisotropic material.

5. Break given input wave into characteristic waves at the beginning of the material

6. Propagate each characteristic wave independently and combine them at the output (end of anistropic material)

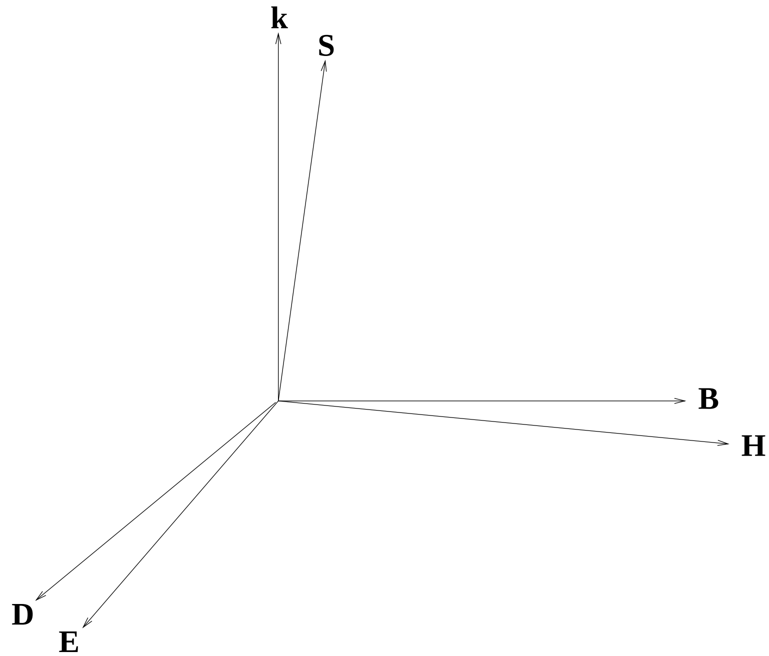

Figure 3.2: Orthogonality relations in a general anisotropic medium (both permittivity and permeability are tensors).

3.2.2 Uniaxial Media

Consider a uniaxial medium with

$$\bar{\bar{\epsilon}} = \begin{bmatrix} \epsilon_e & 0 & 0 \\ 0 & \epsilon_o & 0 \\ 0 & 0 & \epsilon_o \end{bmatrix} \tag{3.27}$$

The permittivity is ϵ_e, called the extraordinary permittivity, along one axis, and ϵ_o, called the ordinary permittivity, along the remaining two. The axis perpendicular to the plane containing the ordinary permittivities is called the *optic axis*. Here is a list of some uniaxial materials with their principal indices of refraction at $\lambda = 633nm$.

Material	$\mathbf{n_o}$	$\mathbf{n_e}$
Calcite	1.6558	1.4852
Lead Molybdate	2.3876	2.2629
Lithium Niobate	2.885	2.2014
Crystalline Quartz	1.5427	1.5518
Ferroelectric Liquid Crystal	1.4467	1.6015

Table 3.1: List of uniaxial materials

Calcite is a very common uniaxial material because it has a fairly large $n_o - n_e$ and is not too fragile or expensive.

Similar to what was done when analyzing plane wave propagation in isotropic materials, we will combine Maxwell's equations to get an equation just in terms of $\overline{E}$. We will plug $\nabla \times \overline{E} = -j\omega\mu\overline{H}$ into $\nabla \times \overline{H} = j\omega\overline{D}$ to get

$$\nabla \times \left(\nabla \times \overline{E}\right) = \nabla\left(\nabla \cdot \overline{E}\right) - \nabla^2\overline{E} = \omega^2\mu\overline{D}. \tag{3.28}$$

Does the term $\nabla \cdot \overline{E}$ still equal zero? The $\partial/\partial x$ and $\partial/\partial y$ terms are both zero because of the plane wave approximation. Since we assume that the wave is traveling down one of the principal axes, $D_z = \epsilon_o E_z$ and

$$\frac{\partial}{\partial z} D_z = \epsilon_o \frac{\partial}{\partial z} E_z = 0 \tag{3.29}$$

so $\nabla \cdot \overline{E} = 0$. We now have the wave equation as given by

$$\nabla^2 \overline{E} + \omega^2 \mu \overline{D} = 0. \tag{3.30}$$

In the isotropic case we then substituted $\overline{D} = \epsilon \overline{E}$ to get the equation entirely in terms of $\overline{E}$. In this case we have to break the wave equation up into the two components as given by

$$\left[\frac{\partial^2}{\partial z^2} + \omega^2 \mu \epsilon_e\right] E_x = 0 \tag{3.31}$$

$$\left[\frac{\partial^2}{\partial z^2} + \omega^2 \mu \epsilon_o\right] E_y = 0 \tag{3.32}$$

The solutions to the wave equations are written in phasor form as

$$E_x = E_{xo} e^{-j\omega\sqrt{\mu\epsilon_e}} \tag{3.33}$$

$$E_y = E_{yo} e^{-j\omega\sqrt{\mu\epsilon_o}} \tag{3.34}$$

These equations lead to the phase velocities and wavenumbers

$$v_e = \frac{1}{\sqrt{\mu\epsilon_e}} \qquad v_o = \frac{1}{\sqrt{\mu\epsilon_o}}$$

$$k_e = \frac{\omega}{v_e} = \omega\sqrt{\mu\epsilon_e} \qquad k_o = \frac{\omega}{v_o} = \omega\sqrt{\mu\epsilon_o}$$

Notice that there are two different allowed phase velocities, depending on the direction of $\overline{E}$! The two possible solutions are *characteristic waves.*

We can think of the uniaxial material as having different indices of refraction depending on the direction of the electric field relative to the optic axis of the crystal. Remember that in this class we assume that the propagation direction is down one of the primary axes. The direction of these primary axes depend on the symmetry axes of the crystal.

If the electric field is perpendicular to the optic axis then the index of refraction is n_o and if the electric field is parallel to the optic axis then the index of refraction is n_e.

Let's look at an example using uniaxial materials to build a prism. Figure 3.3 shows a plane wave traveling through a Rochon prism. The Rochon prism has two regions in which the crystal is rotated to have a different optic axis (O.A.) direction.

1. The incident wave is divided up into its characteristic fields. These fields need to be either parallel or perpendicular to the optic axis. In this case the two fields are (1) an in-plane field with the electric field up and down within the page (shown by an arrow) or (2) an out-of-plane field with the electric field in and out of the page (shown by a dot).

2. In the first region both of the fields are perpendicular to the optic axis. Thus, the index of refraction for both fields is n_o.

3. In the second region, the in-plane field is still perpendicular to the O.A but the out-of-plane field is now parallel to the O.A.

4. The in-plane field see no difference in index of refraction across the boundary. So it does not refract and goes straight through.

5. The index of refraction for out-of-plane field changes from n_o on the left side to n_e on the right side. This difference in index of refraction causes the wave to refract away from the normal (because $n_e < n_o$).

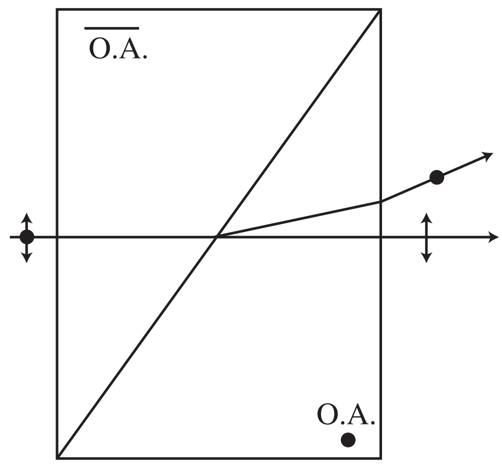

Figure 3.3: A Rochon calcite ($n_e < n_o$) prism.

3.3 Polarization

Polarization refers to the orientation of the electric field vector. In general, a plane wave can be written as

$$\overline{E} = (E_x\hat{x} + E_y\hat{y})e^{-jkz} \tag{3.35}$$

3.3.1 Calculating Polarization

1. multiply by $e^{j\omega t}$
2. convert to time domain (take the real part)
3. plot the electric field as a function of time

3.3.2 Linear Polarization

1. $E_x \neq 0$, $E_y = 0$: $\hat{x}$-polarized
2. $E_x = 0$, $E_y \neq 0$: $\hat{y}$-polarized
3. $E_y = \alpha E_x$ where α is a real constant. Then

   $$\overline{E} = E_x(\hat{x} + \alpha\hat{y})e^{-jkz} \tag{3.36}$$

   The electric field vector makes an angle

   $$\phi = \tan^{-1}\frac{E_y}{E_x} = \tan^{-1}\alpha \tag{3.37}$$

   with the x axis.

Examples:

$$\overline{E} = (3\hat{x} + 4\hat{y})e^{-jkz} \tag{3.38}$$
$$\overline{E} = (j\hat{x} - 3j\hat{y})e^{-jkz} \tag{3.39}$$

3.3.3 Circular Polarization

Suppose $E_y = -jE_x$, E_x real. Then

$$\overline{E} = (E_x\hat{x} - jE_x\hat{y})e^{-jkz} = E_x(\hat{x} - j\hat{y})e^{-jkz} \tag{3.40}$$

In the time domain

$$\begin{aligned}
\overline{\mathcal{E}}(z,t) = \text{Re}\left\{\overline{E}e^{j\omega t}\right\} &= \text{Re}\left\{E_x(\hat{x} + \hat{y}e^{-j\pi/2})e^{-jkz}e^{j\omega t}\right\} && (3.41)\\
&= E_x\text{Re}\left\{\hat{x}e^{j(\omega t - kz)} + \hat{y}e^{j(\omega t - kz - \pi/2)}\right\} && (3.42)\\
&= E_x\left\{\hat{x}\cos(\omega t - kz) + \hat{y}\cos(\omega t - kz - \pi/2)\right\} && (3.43)\\
&= E_x\left\{\hat{x}\cos(\omega t - kz) + \hat{y}\sin(\omega t - kz)\right\} && (3.44)
\end{aligned}$$

At $z = 0$: $\overline{\mathcal{E}}(0,t) = E_x \left\{\hat{x}\cos(\omega t) + \hat{y}\sin(\omega t)\right\}$. We plot the vector as a function of time.

$\omega \mathbf{t}$	$\mathbf{E_x}$	$\mathbf{E_y}$
0	1	0
$\pi/4$	0.707	0.707
$\pi/2$	0	1
$3\pi/4$	-0.707	0.707

Put your thumb in the direction of propagation, and your fingers should point in the direction of rotation. In this case, we get a Right Hand Circularly Polarized (RHCP) wave.

If $E_y = +jE_x$, we get a Left Hand Circularly Polarized (LHCP) wave.

3.3.4 Elliptical Polarization

If $|E_x| \neq |E_y|$ and/or the phase between the two polarizations is not $m\pi/2$, m an integer, then the electric field vector will trace out an ellipse in time. This is elliptical polarization.

Suppose $\overline{E} = \left(E_o\hat{x} + E_o e^{j\frac{\pi}{4}}\hat{y}\right)e^{-jkz}$ then $\overline{E}(t) = \hat{x}E_o\cos(\omega t - kz) + \hat{y}E_o\cos\left(\omega t - kz + \frac{\pi}{4}\right)$

$\omega \mathbf{t}$	$\mathbf{E_x}$	$\mathbf{E_y}$
0	1	0.707
$\pi/4$	0.707	0
$\pi/2$	0	-0.707
$3\pi/4$	-0.707	-1
π	-1	-0.707

The bottom line is: If it is neither linear nor circular, it is elliptical.

Phase Retardation

Suppose we have a slab of uniaxial material. Figure 3.4 shows that since the incident angle is normal there is no refraction. For the first case there is no difference between the two principal fields since there is no difference in their index of refraction. In the second case there is a difference in the index of refraction. This second case results in phase retardation.

Suppose we have a plane wave with $\overline{E}(z=0) = E_o(\hat{x}+\hat{y})$ at the beginning of a uniaxial medium in the region $0 \leq z \leq d$ (Fig. 3.4b). In the medium, it will travel as

$$\overline{E}(z) = E_o(\hat{x}e^{-jk_e z} + \hat{y}e^{-jk_o z}) \tag{3.45}$$

or

$$\overline{E}(z) = E_o(\hat{x}e^{-jn_e k_o z} + \hat{y}e^{-jn_o k_o z}) \tag{3.46}$$

The field at a distance d can be written as

$$\begin{aligned}
\overline{E}(d) &= E_o(\hat{x}e^{-jk_e d} + \hat{y}e^{-jk_o d}) && (3.47)\\
&= E_o e^{-jk_e d}[\hat{x} + \hat{y}e^{j(k_e - k_o)d}] && (3.48)\\
&= E_o e^{-jk_e d}[\hat{x} + \alpha\hat{y}], && (3.49)
\end{aligned}$$

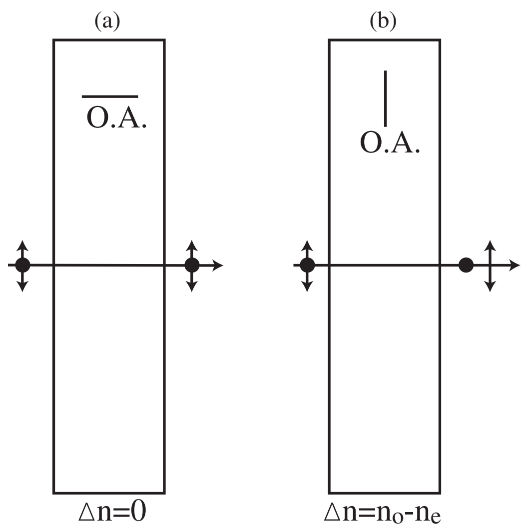

Figure 3.4: A slab of uniaxial crystal.

where

$$\alpha = e^{j\Delta\phi} \tag{3.50}$$

and $\Delta\phi$ is the relative phase shift difference between the two fields as given by $\Delta\phi = (n_o - n_e)\, k_o d$. The term $e^{-jk_e d}$ simply represents a bulk phase shift.

Let's look at a few special cases:

1. **Quarter-Wave Plate:** Let $\Delta\phi = (m + 1/2)\pi$. Then

$$\alpha = e^{j(m+1/2)\pi} = \pm e^{j\pi/2} = \pm j \tag{3.51}$$

$$\overline{E}(d) = E_o e^{-jk^e d}[\hat{x} \pm j\hat{y}] \tag{3.52}$$

which is a circularly polarized wave. Note that the thickness required to obtain this behavior is given as

$$d = \frac{(m + 1/2)\pi}{k_o \Delta n} \tag{3.53}$$

What happens if the Quarter Wave Plate is rotated relative to the initial polarization direction? The electric field becomes

$$\overline{E} = E_o[\sin\theta\hat{x} \pm j\cos\theta\hat{y}] \tag{3.54}$$

This is elliptical polarization with major and minor axes aligned with the initial $\hat{x}$ and $\hat{y}$.

2. **Half-Wave Plate:** Let $(\Delta\phi = (2m + 1)\pi$. Then

$$\alpha = e^{j(2m\pi+\pi)} = -1 \tag{3.55}$$

$$\overline{E}(d) = E_o e^{-jk^e d}[\hat{x} - \hat{y}] \tag{3.56}$$

which is a linearly polarized wave, but is perpendicular to the input wave. Note that the thickness required to obtain this behavior is given as

$$d = \frac{(2n+1)\pi}{k_o \Delta n} \tag{3.57}$$

What happens if the Half Wave Plate is rotated relative to the initial polarization direction? The electric field becomes

$$\begin{aligned} \overline{E}(z) &= E_o(\hat{x}\cos\phi e^{-jk^e z} + \hat{y}\sin\phi e^{-jk^o z}) & (3.58) \\ \overline{E}(d) &= E_o e^{-jk^e d}[\hat{x}\cos\phi + \hat{y}\sin\phi e^{j(k^e - k^o)d}] & (3.59) \\ &= E_o e^{-jk^e d}[\hat{x}\cos\phi - \hat{y}\sin\phi] & (3.60) \end{aligned}$$

So, at $z = 0$ the electric field made an angle ϕ with the positive x axis. After passing through the waveplate, the electric field makes an angle $-\phi$ with the positive x axis.

So the Half Wave Plate can be used to rotate any linear polarization to any other linear polarization.

3. What about for an arbitrary retardation $\Delta\phi$? If the incident electric field is linearly polarized at $45°$ ($\overline{E} = \hat{x} + \hat{y}$), then the transmitted electric field becomes

$$\overline{E} = \left(\hat{x} + e^{j\Delta\phi}\hat{y}\right) \tag{3.61}$$

Figure 3.5 shows the polarization state as a function of $\Delta\phi$.

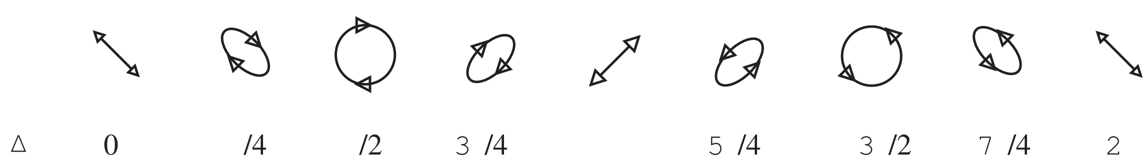

Figure 3.5: polarization state as a function of retardation ($\Delta\phi$).

Applications

What are some possible applications?

1. Variable Ratio Beam Splitter

   Vary HWP angle + polarizing beam splitting cube

2. Intensity Modulator

   Vary retardation + linear polarizer

   Vary HWP rotation angle + linear polarizer

3. Liquid Crystal Display: Applied voltage causes linear polarization to rotate by $90°$. Put LC between polarizers to make an intensity modulator.

3.4 Gytrotropic Media

The most general form of the permittivity tensor is not symetric $\bar{\bar{\epsilon}} = \bar{\bar{\epsilon}}^T$, but it is hermitian $\bar{\bar{\epsilon}} = (\bar{\bar{\epsilon}}^T)^*$. Since the tensor is not symmetric it cannot be made into a diagonal matrix. When the permittivity or permeability tensor is hermitian but not symmetric material is called gyrotropic.

Some gyrotropic materials are:

1. A plasma, which is a neutral mixture of free charge (typically an ionized gas). The electrons have been separated from their nuclei.
2. A crystal with helical or screw-like molecules.
3. A material placed in a DC magnetic field. The strength of the off diagonal terms depend on the type of material and strength of the magnetic field. This effect is called the magneto-optical effect or Faraday rotation.
4. There are also gyromagnetic media such as ferrite in which the tensor is in terms of μ rather than ϵ $\big(\mu_r = \bar{\bar{\mu}}$ and $\epsilon_r = 1\big)$.

We will be concentrating on materials placed in a DC magnetic field; however, the analysis could be applied to the other types of material. The magnetic

For an applied magnetic field $\overline{B}_o = \hat{z}B_o$, the permittivity tensor assumes the form

$$\bar{\bar{\epsilon}} = \epsilon_o \begin{bmatrix} \epsilon_x & -j\kappa & 0 \\ +j\kappa & \epsilon_y & 0 \\ 0 & 0 & \epsilon_z \end{bmatrix}, \tag{3.62}$$

where $\epsilon_o\kappa = \gamma B_o$. Note that the tensor is not symmetric, so it cannot be diagonalized with a coordinate rotation. It is, however, Hermitian, i.e. $\bar{\bar{\epsilon}} = (\bar{\bar{\epsilon}}^T)^*$. We will be assuming that the material is isotropic $(\epsilon_x = \epsilon_y = \epsilon_z = \epsilon)$.

Let's now examine what happens to a plane wave in the medium.

1. We again assume that $\overline{k} = k\hat{z}$ so that $\overline{E} = (\hat{x}E_x + \hat{y}E_y)e^{-jkz}$.

$$\begin{aligned} \nabla \times \overline{E} &= -j\omega\mu\overline{H} \\ -jk\hat{z} \times \overline{E} &= -j\omega\mu\overline{H} \\ -\hat{x}E_y + \hat{y}E_x &= \frac{\omega\mu}{k}(\hat{x}H_x + \hat{y}H_y) \\ E_x &= \frac{\omega\mu}{k}H_y \\ E_y &= -\frac{\omega\mu}{k}H_x \end{aligned} \qquad \begin{aligned} \nabla \times \overline{H} &= j\omega\overline{D} \\ -jk\hat{z} \times \overline{H} &= j\omega\overline{D} \\ \hat{x}H_y - \hat{y}H_x &= \frac{\omega}{k}(\hat{x}D_x + \hat{y}D_y) \\ H_x &= -\frac{\omega}{k}D_y \\ H_y &= \frac{\omega}{k}D_x \end{aligned}$$

2. Place $\overline{H}$ into the Ev equation

$$
\begin{aligned}
E_x &= \left(\frac{\omega\mu}{k}\right)\left(\frac{\omega}{k}\right)D_x \\
E_y &= \left(\frac{\omega\mu}{k}\right)\left(\frac{\omega}{k}\right)D_y
\end{aligned}
$$

3. $\overline{D}$ is related to $\overline{E}$ by the permittivity matrix

$$\overline{D} = \overline{\overline{\epsilon}}\,\overline{E} \tag{3.63}$$

(3.64)

$$\begin{bmatrix} D_x \\ D_y \\ D_z \end{bmatrix} = \epsilon_o \begin{bmatrix} \epsilon & -j\kappa & 0 \\ +j\kappa & \epsilon & 0 \\ 0 & 0 & \epsilon \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix} \tag{3.65}$$

$$D_x = \epsilon_o\left(\epsilon E_x - j\kappa E_y\right) \tag{3.66}$$

$$D_y = \epsilon_o\left(j\kappa E_x + \epsilon E_y\right) \tag{3.67}$$

4. Substitute in $\overline{D}$ to get equation just in terms of Ev

$$E_x = \mu_o\epsilon_o\left(\frac{\omega}{k}\right)^2\left(\epsilon E_x - j\kappa E_y\right) \tag{3.68}$$

$$E_y = \mu_o\epsilon_o\left(\frac{\omega}{k}\right)^2\left(j\kappa E_x + \epsilon E_y\right) \tag{3.69}$$

5. Rearranging these two equations gives

$$E_x\left(1-\left(\frac{\omega}{kc}\right)^2\epsilon\right) = -j\left(\frac{\omega}{kc}\right)^2\kappa E_y \qquad E_y\left(1-\left(\frac{\omega}{kc}\right)^2\epsilon\right) = j\left(\frac{\omega}{kc}\right)^2\kappa E_x$$

$$E_y = \frac{1-\left(\frac{\omega}{kc}\right)^2\epsilon}{-j\left(\frac{\omega}{kc}\right)^2\kappa}E_x \qquad E_y = \frac{j\left(\frac{\omega}{kc}\right)^2\kappa}{1-\left(\frac{\omega}{kc}\right)^2\epsilon}E_x$$

6. Combining these two equations gives

$$\frac{1-\left(\frac{\omega}{kc}\right)^2\epsilon}{-j\left(\frac{\omega}{kc}\right)^2\kappa} = \frac{j\left(\frac{\omega}{kc}\right)^2\kappa}{1-\left(\frac{\omega}{kc}\right)^2\epsilon}$$

$$1-\left(\frac{\omega}{kc}\right)^2\epsilon = \pm\left(\frac{\omega}{kc}\right)^2\kappa$$

$$1 = \left(\frac{\omega}{kc}\right)^2 (\epsilon \pm \kappa)$$

So the wavevectors are

$$k = \frac{\omega}{c}\sqrt{\epsilon \pm \kappa}$$

and the corresponding velocity $\left(v = \frac{\omega}{k}\right)$ is

$$v = \frac{c}{\sqrt{\epsilon \pm \kappa}}$$

So there are two possible wavevector for the two principal electric fields. Each of these electric fields propagate with a different velocity $\left(v = \frac{c}{n}\right)$. These fields exhibit a different index of refraction. Let's call these two principle indices of refraction

$$n^+ = \sqrt{\epsilon + \kappa} \tag{3.70}$$
$$n^- = \sqrt{\epsilon - \kappa} \tag{3.71}$$

7. Now that we know the speed of the principal electric fields, we need to know the form of these principle fields. We will plug k back into the equation for the electric field. This equation has the term $\left(\frac{\omega}{kc}\right)^2$, which is equal to

$$\begin{aligned} n^+ &= \sqrt{\epsilon + \kappa} \\ \left(\frac{\omega}{kc}\right)^2 &= \left(\frac{\omega}{c}\right)^2 \frac{c^2}{\omega^2 (\epsilon + \kappa)} \\ &= \frac{1}{\epsilon + \kappa} \end{aligned} \qquad \begin{aligned} n^- &= \sqrt{\epsilon - \kappa} \\ \left(\frac{\omega}{kc}\right)^2 &= \left(\frac{\omega}{c}\right)^2 \frac{c^2}{\omega^2 (\epsilon - \kappa)} \\ &= \frac{1}{\epsilon - \kappa} \end{aligned}$$

The electric field is given by

$$E_y = \frac{1 - \left(\frac{\omega}{kc}\right)^2 \epsilon}{-j\left(\frac{\omega}{kc}\right)^2 \kappa} E_x,$$

which is equal to

$$\begin{aligned} E_y &= \frac{1 - \frac{\epsilon}{\epsilon+\kappa}}{-j\frac{\kappa}{\epsilon+\kappa}} E_x \\ &= \frac{\epsilon + \kappa - \epsilon}{-j\kappa} E_x \\ &= \frac{\kappa}{-j\kappa} E_x \\ &= jE_x \end{aligned} \qquad \begin{aligned} E_y &= \frac{1 - \frac{\epsilon}{\epsilon-\kappa}}{-j\frac{\kappa}{\epsilon-\kappa}} E_x \\ &= \frac{\epsilon - \kappa - \epsilon}{-j\kappa} E_x \\ &= \frac{-\kappa}{-j\kappa} E_x \\ &= -jE_x \end{aligned}$$

So, the electric fields are of the form

$$\overline{E}^+ = E_o(\hat{x} + j\hat{y})e^{-jk^+ z}. \qquad \overline{E}^- = E_o(\hat{x} - j\hat{y})e^{-jk^- z}.$$

The $+$ sign corresponds to LH circular polarization and the $-$ sign to RH circular polarization. We will rename the $+$ term to L and the $-$ term to R signifying the polarization state of the principle fields as given by

$$n^+ \Longrightarrow n^L = \sqrt{\epsilon + \kappa} \tag{3.72}$$

$$n^- \Longrightarrow n^R = \sqrt{\epsilon - \kappa} \tag{3.73}$$

$$\tag{3.74}$$

So the left handed polarization exhibits an index of refraction of n^L and the right handed polarization exhibits an index of refraction of n^R.

8. To analyze the propagation of an arbitrary plane wave through the the gyrotropic material, we need to (1) express it in terms of RHCP and LHCP waves. (2) Propagate the principle fields through the material. (3) recombine to see what the new field is.

   The incident plane wave will be expressed in terms of x and y components, even if it has linear, elliptical, or circular polarization. So we need to express linear polarizations in terms of circular polarizations. To do this, we write

$$E_x\hat{x} + E_y\hat{y} = E_R(\hat{x} - j\hat{y}) + E_L(\hat{x} + j\hat{y}) \tag{3.75}$$

   which after equating vector components leads to

$$E_x = E_R + E_L \tag{3.76}$$

$$E_y = -j(E_R - E_L) \tag{3.77}$$

   Solving (3.76) for E_L and placing into (3.77) lead to

$$E_y = -j(E_R - E_x + E_R) = -j(2E_R - E_x) \tag{3.78}$$

$$E_R = \frac{E_x + jE_y}{2} \tag{3.79}$$

$$E_L = E_x - E_R = \frac{E_x - jE_y}{2} \tag{3.80}$$

   Let's then start with an easy plane wave with

$$\overline{E}(z=0) = E_o\hat{x} = \frac{E_o}{2}(\hat{x} - j\hat{y}) + \frac{E_o}{2}(\hat{x} + j\hat{y}), \tag{3.81}$$

   which has components

$$E_R = \frac{E_o}{2} \tag{3.82}$$

$$E_L = \frac{E_o}{2} \tag{3.83}$$

   and will travel as

$$\overline{E}(z) = \frac{E_o}{2}(\hat{x} - j\hat{y})e^{-jk_Rz} + \frac{E_o}{2}(\hat{x} + j\hat{y})e^{-jk_Lz} \tag{3.84}$$

   Combine the x and y components to get

$$\overline{E}(z) = \frac{E_o}{2}\left\{\hat{x}(e^{-jk_Rz} + e^{-jk_Lz}) - j\hat{y}(e^{-jk_Rz} - e^{-jk_Lz})\right\} \tag{3.85}$$

factor out the average phase term to get

$$\overline{E}(z) = \frac{E_o}{2} e^{-jk_R z/2} e^{-jk_L z/2} \tag{3.86}$$

$$\left\{ \hat{x} \left[e^{-j(k_R-k_L)z/2} + e^{j(k_R-k_L)z/2} \right] - j\hat{y} \left[e^{-j(k_R-k_L)z/2} - e^{j(k_R-k_L)z/2} \right] \right\} \tag{3.87}$$

$$= \frac{E_o}{2} e^{-j(k_R+K_L)z/2} \left\{ 2\hat{x} \cos\left[\frac{(k_R-k_L)z}{2} \right] - j(-j2)\hat{y} \sin\left[\frac{(k_R-k_L)z}{2} \right] \right\} \tag{3.88}$$

$$= E_o e^{-j(k_R+K_L)z/2} \left\{ \hat{x} \cos\left[\frac{(k_R-k_L)z}{2} \right] - \hat{y} \sin\left[\frac{(k_R-k_L)z}{2} \right] \right\} \tag{3.89}$$

This polarization is still linear because the x and y components are still in phase. However, the linear polarization axis changes with respect to the x-axis as

$$\phi = \tan^{-1}\left\{ \frac{E_y}{E_x} \right\} \tag{3.90}$$

$$= \tan^{-1}\left\{ \frac{-\sin[(k_R-k_L)z/2]}{\cos[(k_R-k_L)z/2]} \right\} \tag{3.91}$$

$$= \tan^{-1}\left\{ -\tan[(k_R-k_L)z/2] \right\} \tag{3.92}$$

$$= -\frac{(k_R-k_L)z}{2} \tag{3.93}$$

$$= -\frac{k_o}{2}\left(n^R - n^L \right) z \tag{3.94}$$

So, if we view the wave from behind (we look in the z direction), the electric field rotates counter-clockwise as it passes through the medium. This phenomenon is known as Faraday rotation.

Now, for a wave traveling in the $-z$ direction, repetition of the entire analysis shows that the polarization will rotate clockwise when looking from behind (we look in the $-z$ direction). When looking at the wave as it approaches (in the z direction), however, this is a counter-clockwise rotation. So, the behavior is non-reciprocal. This property is very important in some applications, especially when we want different behaviors for forward and reverse traveling waves.

Optically active materials (such as some sugar-water solutions) also rotate polarization, but in a reciprocal way—if the wave were to reflect at the end of the medium, it would rotate back to its original polarization.

3.4.1 Isolator

In a laser system, often we to eliminate any power being reflected back into the laser. In order to prevent this reflected wave, we would like to have an isolator, or a device which allows propagation in one direction and attenuates waves traveling in the opposite direction.

This can be accomplished by sandwiching a section of gyrotropic material between two polarizing filters of some kind. The allowed polarization of the second filter is rotated by 45°, and the length of the gyrotropic material is such that polarization is rotated by 45°. In the forward direction, the allowed polarization is at, say 0°. The polarization rotates counterclockwise (viewed from behind) to 45° and passes through the second filter. In the reverse direction, the alllowed polarization starts at 45° and rotates counterclockwise (viewed as it approaches) to 90°, and cannot pass through the first filter (Fig. **??**).

In an optical isolator, the material is a magneto-optical material. Most materials have some amount of magneto-optical characteristic. However, it is fairly small for most matarials.

The magneto-optical parameter usually used to characterize a material is the Verdet constant. The Verdet constant for most optically transparent materials typically decrease with wavelength. Here are the Verdet constant for some materials

A small magneto-optical coefficient means that the Faraday rotator requires either a very large applied magnetic field or a large interaction length. A commonly used crystal is Terbium Gallium Garnet (TGG) that has an magneto-optical coefficient 40x larger than that of glass.

3.4.2 Gyromagnetic Media

In a microwave isolator, the material used is gyromagnetic, so that the permeability becomes a matrix with the same form as (3.62). Such a material is obtained by placing a ferrite (magnetic medium) in a DC magnetic field (biased ferrite), so that the magnetic moments of the atoms in the ferrite line up. When an AC magnetic field is applied, the magnetic moments are perturbed, leading to an additional magnetic field which is analogous to the $\overline{D}$, $\overline{E}$ relationship for a biased plasma.

In a ferrite, the magnetization $\overline{M}$ roughly obeys the relation

$$\frac{\partial \overline{M}}{\partial t} = g\mu_o \overline{M} \times \overline{H}, \tag{3.95}$$

where g is called the *gyromagnetic ratio*.

After some derivation the magnetic flux can be derived to be related to the magnetic field by $\overline{B} = \overline{\overline{\mu}}\overline{H}$, where the permeability tensor is

$$\overline{\overline{\mu}} = \begin{bmatrix} \mu & j\mu_g & 0 \\ -j\mu_g & \mu & 0 \\ 0 & 0 & \mu_z \end{bmatrix}, \tag{3.96}$$

and

$$\mu = \mu_o \left[1 - \frac{\left(g\mu_o/\omega\right)^2 H_o M_o}{1 - \left(g\mu_o H_o/\omega\right)^2} \right] \tag{3.97}$$

$$\mu_g = \mu_o \left[\frac{\left(g\mu_o M_o/\omega\right)}{1 - \left(g\mu_o H_o/\omega\right)^2} \right] \tag{3.98}$$

$$\mu_z = \mu_o \left[1 + \frac{H_o}{M_o} \right]. \tag{3.99}$$

Recall that

$$B = \mu_o \mu_r = \mu_o \left(H_o + M_o\right). \tag{3.100}$$

So,

$$\mu_r H = H_+ M \tag{3.101}$$

$$M = H\left(\mu_r - 1\right). \tag{3.102}$$

For the permeability tensor, there is a large externally applied magnetic field B_o. Therefore, the M_o and H_o terms in permeability matrix are given by

$$\begin{aligned} H_o &= \frac{B_o}{\mu_o \mu_r} && (3.103) \\ M_o &= \frac{B_o}{\mu_o \mu_r}(\mu_r - 1) \approx \frac{B_o}{\mu_o} && (3.104) \\ & && (3.105) \end{aligned}$$

3.4.3 Optical Faraday Media

Optical materials are typically characterized using the Verdet constant V. Ther Verdet constant is used to characterize the rotary power of the material ρ and is given by

$$\rho = \frac{\pi\left(n^- - n^+\right)}{\lambda_o} = VB, \qquad (3.106)$$

where B is the externally DC magnetic flux applied in the z direction and V is the verdet constant. The verdet constant for some commonly used materials is V=0.0124 min/cm-Oe for fused silica are V=-0.46 min/G-cm for Terbium Gallium Garnet.